Floating-point numbers

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Integration of mpfr, mpfi, mpc, fplll and cxsc in GAP

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Abstract

This document describes the package Float, which implements in GAP arbitrary-precision floating-point numbers.

For comments or questions on Float please contact the author.

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Float package

2.1 A sample run

The extended floating-point capabilities of GAP are installed by loading the package via `LoadPackage("float");` and selecting new floating-point handlers via `SetFloats(MPFR)`, `SetFloats(MPFI)`, `SetFloats(MPC)` or`SetFloats(CXSC)`, depending on whether high-precision real, interval or complex arithmetic are desired, or whether a fast package containing all four real/complex element/interval arithmetic is desired:

Example

```
gap> LoadPackage("float");
Loading FLOAT 0.7.0 ...
true
gap> SetFloats(MPFR); # floating-point
gap> x := 4*Atan(1.0);
.314159e1
gap> Sin(x);
.169569e-30
gap> SetFloats(MPFR,1000); # 1000 bits
gap> x := 4*Atan(1.0);
.314159e1
gap> Sin(x);
.125154e-300
gap> String(x,300);
".314159265358979323846264338327950288419716939937510582097494459230781640628 6\
208998628034825342117067982148086513282306647093844609550582231725359408128481\
117450284102701938521105559644622948954930381964428810975665933446128475648233\
786783165271201909145648566923460348610454326648213393607260249141 27e1"
gap>
gap> SetFloats(MPFI); # intervals
gap> x := 4*Atan(1.0);
.314159e1(99)
gap> AbsoluteDiameter(x); Sup(x); Inf(x);
.100441e-29
.314159e1
.314159e1
gap> Sin(x);
-.140815e-29(97)
gap> 0.0 in last;
```

```
true
gap> 1.0; # exact representation
.1e1(inf)
gap> IncreaseInterval(last,0.001); # now only 8 significant bits
.1e1(8)
gap> IncreaseInterval(last,-0.002); # now becomes empty
\emptyset
gap> r2 := Sqrt(2.0);
.141421e1(99)
gap> MinimalPolynomial(Rationals,r2);
-2*x_1^2+1
gap> Cyc(r2);
E(8)-E(8)^3
gap>
gap> SetFloats(MPC); # complex numbers
gap> z := 5.0-1.0i;
.5e1-.1e1i
gap> (1+1.0i)*last^4*(239+1.0i);
.228488e6
gap> Exp(6.2835i);
.1e1+.314693e-3i
```

Chapter 3

Polynomials

3.1 The Floats pseudo-field

Polynomials with floating-point coefficients may be manipulated in GAP; though they behave, in subtle ways, quite differently than polynomials over rings. A "pseudo-field" of floating-point numbers is available to create them using the standard GAP syntax.

3.1.1 FLOAT_PSEUDOFIELD

▷ FLOAT_PSEUDOFIELD (global variable)

The "pseudo-field" of floating-point numbers, containing all floating-point numbers in the current implementation.

Note that it is not really a field, e.g. because addition of floating-point numbers is not associative. It is mainly used to create indeterminates, as in the following example:

Example
```
gap> x := Indeterminate(FLOAT_PSEUDOFIELD,"x");
x
gap> 2*x^2+3;
2.0*x^2+3.0
gap> Value(last,10);
203.0
```

3.2 Roots of polynomials

The Jenkins-Traub algorithm has been implemented, in arbitrary precision for MPFR and MPC.

Furthermore, CXSC can provide complex enclosures for the roots of a complex polynomial.

3.3 Finding integer relations

The PSLQ algorithm has been implemented by Steve A. Linton, as an external contribution to Float. This algorithm receives as input a vector of floats x and a required precision ε, and seeks an integer vector v such that $|x \cdot v| < \varepsilon$. The implementation follows quite closely the original article [BB01].

3.3.1 PSLQ

▷ `PSLQ(x, epsilon[, gamma])` (function)

▷ `PSLQ_MP(x, epsilon[, gamma[, beta]])` (function)

Returns: An integer vector v with $|x \cdot v| < \varepsilon$.

The PSLQ algorithm by Bailey and Broadhurst (see [BB01]) searches for an integer relation between the entries in x.

β and γ are algorithm tuning parameters, and default to $4/10$ and $2/\sqrt(3)$ respectively.

The second form implements the "Multi-pair" variant of the algorithm, which is better suited to parallelization.

Example

```
gap> PSLQ([1.0,(1+Sqrt(5.0))/2],1.e-2);
[ 55, -34 ] # Fibonacci numbers
gap> RootsFloat([1,-4,2]*1.0);
[ 0.292893, 1.70711 ] # roots of 2x^2-4x+1
gap> PSLQ(List([0..2],i->last[1]^i),1.e-7);
[ 1, -4, 2 ] # a degree-2 polynomial fitting well
```

3.4 LLL lattice reduction

A faster implementation of the LLL lattice reduction algorithm has also been implemented. It is accessible via the commands `FPLLLReducedBasis(m)` and `FPLLLShortestVector(m)`.

Chapter 4

Implemented packages

4.1 MPFR

4.1.1 IsMPFRFloat

▷ `IsMPFRFloat` (filter)
▷ `TYPE_MPFR` (global variable)

The category of floating-point numbers.
Note that they are treated as commutative and scalar, but are not necessarily associative.

4.2 MPFI

4.2.1 IsMPFIFloat

▷ `IsMPFIFloat` (filter)
▷ `TYPE_MPFI` (global variable)

The category of intervals of floating-point numbers.
Note that they are treated as commutative and scalar, but are not necessarily associative.

4.3 MPC

4.3.1 IsMPCFloat

▷ `IsMPCFloat` (filter)
▷ `TYPE_MPC` (global variable)

The category of intervals of floating-point numbers.
Note that they are treated as commutative and scalar, but are not necessarily associative.

4.4 CXSC

4.4.1 IsCXSCReal

▷ `IsCXSCReal` (filter)
▷ `IsCXSCComplex` (filter)
▷ `IsCXSCInterval` (filter)
▷ `IsCXSCBox` (filter)
▷ `TYPE_CXSC_RP` (global variable)
▷ `TYPE_CXSC_CP` (global variable)
▷ `TYPE_CXSC_RI` (global variable)
▷ `TYPE_CXSC_CI` (global variable)

The category of floating-point numbers.

Note that they are treated as commutative and scalar, but are not necessarily associative.

4.5 FPLLL

4.5.1 FPLLLReducedBasis

▷ `FPLLLReducedBasis(`*m*`)` (operation)

Returns: A matrix spanning the same lattice as *m*.

This function implements the LLL (Lenstra-Lenstra-Lovász) lattice reduction algorithm via the external library fplll.

The result is guaranteed to be optimal up to 1%.

4.5.2 FPLLLShortestVector

▷ `FPLLLShortestVector(`*m*`)` (operation)

Returns: A short vector in the lattice spanned by *m*.

This function implements the LLL (Lenstra-Lenstra-Lovász) lattice reduction algorithm via the external library fplll, and then computes a short vector in this lattice.

The result is guaranteed to be optimal up to 1%.

References

[BB01] D. H. Bailey and D. J. Broadhurst. Parallel integer relation detection: techniques and applications. *Math. Comp.*, 70(236):1719–1736 (electronic), 2001. 7, 8

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